Available online at www.sciencedirect.com
ScienceDirect
Journal of the European Ceramic Society 28 (2008) 133–138
www.elsevier.com/locate/jeurceramsoc

Measurements of complex materials constants of piezoelectric ceramics: Radial vibrational mode of a ceramic disk

Maeda Masaki *, Hiroyuki Hashimoto, Washihira Masahiko, Ikuo Suzuki

Department of Engineering Physics, Electronics and Mechanics, Graduate School of Engieering, Nagoya Institute of Technology, Showa-ku, Nagoya, Aichi 466-8555, Japan

Received 11 February 2007; received in revised form 8 May 2007; accepted 16 May 2007
Available online 6 August 2007

Abstract

An iterative method for characterization of piezoelectric ceramics of a disk in radial vibrational mode has been developed. The complex materials constants (dielectric constants, elastic compliance coefficients and piezoelectric constants) of piezoelectric ceramics with various mechanical quality factor (Q) were accurately determined. An explicit procedure for estimating the appropriate initial values of the complex materials constants for a nonlinear iteration process has been presented. The present nonlinear iterative fitting procedure was carried out for the experimental admittance spectra $Y = G + jB$ for various specimens with moderate (667) or low (29) Q values and the validity of the method was verified.

Keywords: Complex materials constants; Iteration method; Piezoelelctric ceramics; PZT; $KNbO_3$

1. Introduction

In the previous paper, an iterative method for characterization of piezoelectrics with a bar in length extensional vibration mode has been reported.[1] With this method both real and imaginary parts for the whole set of dielectric, elastic and piezoelectric constants in piezoelectric materials have been successfully determined. Though a length extensional vibrational mode of a bar poled and exited in direction of its thickness is simple and fundamental, one of the more frequently used mode is a radial vibration of an axially poled disk due to its relative ease of manufacturing, polarization processes and wide practical applications.

There have been several reports on determination of the complex materials constants for ceramics in a radial vibration mode.[2–4] The methods used are classified as iterative or noniterative. The noninterative method for the radial mode has been proposed by Sherrit et al.[2] In their process nine characteristic frequencies (the first and the second resonance frequencies, $f_s^{(1)}$ and $f_s^{(2)}$, the first antiresonance frequency $f_p^{(1)}$ and their half-band frequencies) should be measured with high accuracy, however accurate measurements of those frequencies will be difficult task especially for the specimens with low mechanical quality factor Q. For radial modes, the noniterative method is valid only for materials with intermediate mechanical quality factors. The iterative method proposed by Alemany et al.[3] can be applied to the specimens with low Q values. However their method requires a judicious choice of the admittance data at the frequencies where the maximum piezoelectric contributions are contained. The criteria for determining those frequencies are not explicit, hence an iteration process is required to find those frequencies. In their method, measurements of new admittance data at the new frequencies are necessary in the iteration process, thus their measurements took time. Recently a new iterative method for determining the dielectric, piezoelectric and elastic constants in a complex form for the piezoelectric ceramics in radial mode has been reported by Amarande et al.[4] They improved Alemany's process by separating the data acquisition and processing. However, their method still need to determine the characteristic frequencies, which are similar to those required by Alemany's method. The explicit procedure for estimating the appropriate initial parameters for the iteration process has not been presented because of the indefinite characteristic frequencies.

A new routine with an explicit procedure for determining the initial values and up-to-date nonlinear least square itera-

* Corresponding author.
E-mail address: maeda@nitech.ac.jp (M. Masaki).

0955-2219/$ – see front matter © 2007 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jeurceramsoc.2007.05.021

tion method are demanded to automate the iteration process for characterization of a ceramic disk in a radial vibration mode.

An iterative automated procedure for determining the complex materials constants from G (conductance) and B (susceptance) spectra of a ceramic disk in the radial vibration mode has been developed in this paper. The details of the procedure will be described in the following sections.

2. Experiment

Specimens used in the experiments were $K_{0.9}Li_{0.1}NbO_3$, $PbZr_{0.51}Ti_{0.49}O_3$ (1 wt%: Nb_2O_5), $PbZr_{0.51}Ti_{0.49}O_3$ and $PbZr_{0.51}Ti_{0.49}O_3$ (0.5 wt%: MnO_2). Ceramic specimens (radius 8 mm, thickness 0.8 mm) were used. The conductance (G) and susceptance (B) of the specimens were measured in the frequency region around the fundamental ($f_s^{(1)}$) and 1st overtone ($f_s^{(2)}$) resonances by an impedance-gain-phase analyzer (Schlunberge SI 1260), controlled by a computer via a GP–IB interface. The fundamental and 1st overtone resonance bands were swept in the frequency range of $10 \times \Delta f_s^{(1)}$ and $10 \times \Delta f_s^{(2)}$, where $\Delta f_s^{(1)}$ and $\Delta f_s^{(2)}$ are the half-band widths of the fundamental and 1st overtone resonance bands, respectively. The $\Delta f_s^{(1)}$ and $\Delta f_s^{(2)}$ are defined as the difference in frequencies between the maximum (f_1) and the minimum (f_2) of B in the fundamental and 1st resonance bands, respectively. The number of data acquisition points in the measured frequency range was 200. The programs for data acquisition and further processing were written in Visual Basic (Microsoft). Data on the frequency dependence of the G and B were stored in a personal computer. The values of G at the resonance frequency $f_s^{(1)}$ ($= f_0$) and G and B at frequencies f_1 and f_2 were picked up from the data of $G(f)$ and $B(f)$.

3. Iteration Procedure

The basic idea for the present nonlinear iteration process is based on the method described for a bar specimen in the length extensional mode.[1] However, in the case of a ceramic disk poled and excited in its thickness in the radial vibration mode, the circular shape impose some differences in mathematical description, thus the algorithm must be modified.

The electrical admittance of a electrically poled disk with a radial vibration mode is obtained from the linear piezoelectric basic equations.[5] The admittance as a function of frequencies is given by the following expression:

$$Y = j\omega\frac{\pi a^2}{t}\frac{2d_{31}^2}{(1-\sigma)s_{11}^E}\frac{(1+\sigma)J_1(z)}{z\,J_0(z)-(1-\sigma)J_1(z)} + \omega\frac{\pi a^2}{t}\left(\varepsilon_{33}^T - \frac{2}{1-\sigma}\frac{d_{31}^2}{s_{11}^E}\right), \quad (1)$$

where J_0 and J_1 are Bessel's functions of the first kind of 0th and 1st orders and

$$z = \omega a\sqrt{\rho s_{11}^E(1-\sigma^2)},$$

where the density ρ, the radius a, the thickness t, Poisson's ratio σ and the driving frequency ω are real quantities. The other symbols refer to complex parameters, such as the piezoelectric charge coefficient d_{31}, the dielectric constant at constant stress ε_{33}^T, and the elastic compliance at constant electric field s_{11}^E. An important parameter obtained from the previous constants is electromechanical coupling factor k_p given by

$$k_p^2 = \frac{2d_{31}^2}{\varepsilon_{33}^T s_{11}^E(1+\sigma)}.$$

The complex materials constants s_{11}^E, ε_{33}^T, and d_{31} are defined as[6]

$$s_{11}^E = s_{11}^{E'} - js_{11}^{E''}, \qquad \varepsilon_{33}^T = \varepsilon_{33}^{T'} - j\varepsilon_{33}^{T''}$$

and

$$d_{31} = d_{31}' - jd_{31}''.$$

The admittance formula given in Eq. (1) is expanded around $s_{11}^{E'}$, $\varepsilon_{33}^{T'}$, and d_{31}' and the terms over the square of $s_{11}^{E''}$, $\varepsilon_{33}^{T''}$ and d_{31}'' are discarded. This simplification leads to the results:

$$Y_1 = G_1 + jB_1$$

where Y_1, G_1 and B_1 represent the linearized admittance, conductance and susceptance, respectively. The expressions for the conductance G_1 and susceptance B_1 are given in Appendix A.

The fundamental resonance frequency ($f_s^{(1)}$) and the 1st overtone frequency ($f_s^{(2)}$) give two values of z as follows:

$$z_1 = 2\pi\, f_s^{(1)} a\sqrt{\rho\, s_{11}^{E'}(1-\sigma^2)},$$

$$z_2 = 2\pi\, f_s^{(2)} a\sqrt{\rho\, s_{11}^{E'}(1-\sigma^2)}.$$

These two z obey the following relation:

$$\frac{z_2}{z_1} = \frac{f_s^{(2)}}{f_s^{(1)}}. \quad (2)$$

At both the fundamental resonance frequency $f_s^{(1)}$ and the 1st overtone frequency $f_s^{(2)}$, the resonance conditions should be satisfied by the following equations:

$$\mathfrak{J}_1(z_1) = 1-\sigma, \quad (3)$$

$$\mathfrak{J}_1(z_2) = 1-\sigma, \quad (4)$$

where $\mathfrak{J}_1(z) = z\,J_0(z)/J_1(z)$ is the modified quotient of cylindrical functions of the first kind. The problem is just to solve a system of two transcendental equations (3) and (4) with two unknown parameters σ and z_1 under the condition of Eq. (2). We calculate the σ and z_1 which satisfies these Eqs. (3) and (4) by a self-consistent iteration process. This is the only place where the 1st overtone frequency $f_s^{(2)}$ takes part. In the rest of the process only experimental data around the fundamental resonance are sufficient.

We can determine the initial values for the iteration process as follows. The real part of elastic compliance $s_{11}^{E'}$ is calculated

from the fundamental resonance frequency $f_s^{(1)}$ ($= f_0$).

$$s_{11}^{E'} = \frac{z_1}{\rho a^2 \omega_1^2 (1-\sigma^2)},$$

where

$$z_1 = 2\pi f_s^{(1)} a \sqrt{\rho s_{11}^{E'}(1-\sigma^2)}, \qquad \omega_1 = 2\pi f_s^{(1)}.$$

The imaginary part $s_{11}^{E''}$ is given as[3]

$$s_{11}^{E''} = s_{11}^{E'} \left(\frac{f_2 - f_1}{f_0} \right).$$

Since $s_{11}^{E'}$ and $s_{11}^{E''}$ are known, the unknown parameters in B are $\varepsilon_{33}^{T'}$ and d'_{31}. The initial values of $\varepsilon_{33}^{T'}$ and d'_{31} can be obtained by solving the simultaneous equations of $B_1(f_1)$ and $B(f_2)$ for $\varepsilon_{33}^{T'}$ and d'_{31}.

$$\begin{pmatrix} B_1(f_1) \\ B_1(f_2) \end{pmatrix} = \begin{pmatrix} b_1(f_1) & b_2(f_1) \\ b_1(f_2) & b_2(f_2) \end{pmatrix} \begin{pmatrix} \varepsilon_{33}^{T'} \\ d'^2_{31} \end{pmatrix} \tag{5}$$

The expressions for $b_1(f)$ and $b_2(f)$ are given in Appendix A.

At this stage, $s_{11}^{E'}$, $s_{11}^{E''}$, $\varepsilon_{33}^{T'}$ and d'_{31} are known and $\varepsilon_{33}^{T''}$ and d''_{31} are the only unknown parameters in G_1. The initial values of $\varepsilon_{33}^{T''}$ and d''_{31} can be determined by solving the simultaneous equations of $G_1(f_1)$ and $G_1(f_2)$ for $\varepsilon_{33}^{T''}$ and d''_{31}.

$$\begin{pmatrix} G_1(f_1) \\ G_1(f_2) \end{pmatrix} = \begin{pmatrix} g_1(f_1) & g_2(f_1) \\ g_1(f_2) & g_2(f_2) \end{pmatrix} \begin{pmatrix} \varepsilon_{33}^{T''} \\ d''_{31} \end{pmatrix} + \begin{pmatrix} g_3(f_1) \\ g_3(f_2) \end{pmatrix} \tag{6}$$

The expressions for the $g_1(f)$, $g_2(f)$ and $g_3(f)$ are given in Appendix A.

The initial values of s_{11}^{E}, ε_{33}^{T} and d_{31} are put into Eq. (1) and the iteration calculation starts to simulate the $G_1(f)$ and $B_1(f)$. In the present nonlinear iteration method, Levenberg–Marquardt's algorithm[7] is employed and the complex materials constants are determined by minimizing χ^2 as has been presented in the previous paper.[1] The determination of the complex materials constants has been achieved by a curve fitting approach. The whole resonance spectra are used and the correctness of the fitting is ensured in the whole measured range. The iteration calculation has been performed with the use of a commercially available program package (IGOR Pro. WaveMetrics Inc.). This program allows complex variables as fitting parameters. The flow chart of the nonlinear iteration process is shown in Fig. 1.

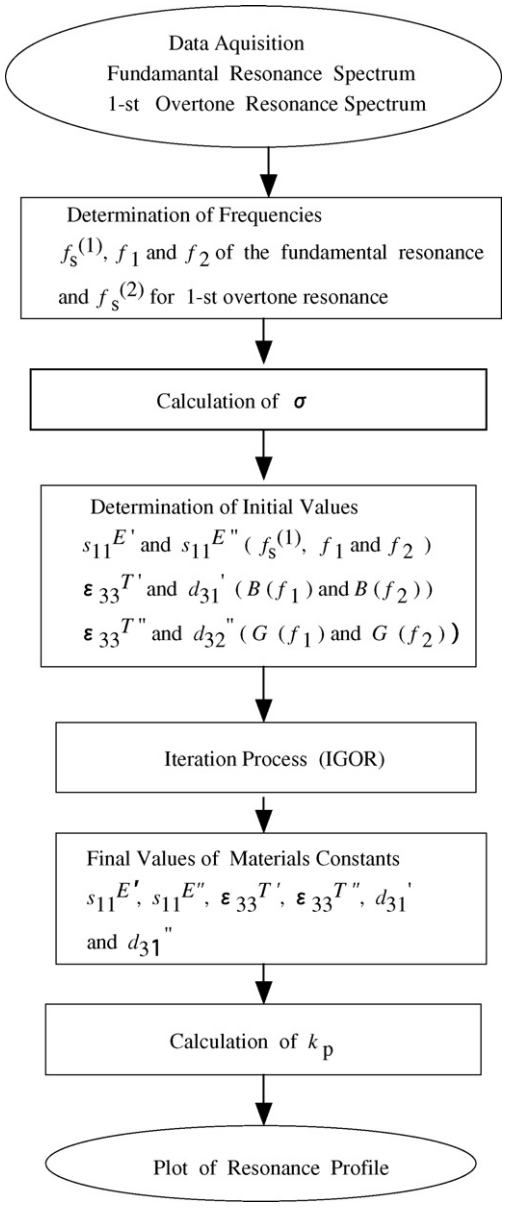

Fig. 1. Blockdiagram of nonlinear iteration process.

4. Results

The results of the simulation for $G(f)$ and $B(f)$ of the specimen $K_{0.9}Li_{0.1}NbO_3$ is shown in Fig. 2(a) and (b). The mechanical Q factor for this specimen is 29 which is much lower comparing with the ordinary ceramic specimens and the signal level of the admittance is less than 1/100 comparing with $PbZr_{0.51}Ti_{0.49}O_3$ (0.5 wt%: MnO_2). The agreement between the experimental G and B spectra and the calculated ones is rather excellent taking into account of the poor resonance characteristics. The complex materials constants obtained in the present iteration method (G–B fitting) are given in Table 1 in addition to the values calculated by the conventional resonance–antiresonance method (R–A method). The results of the simulation for $G(f)$ and $B(f)$ for $PbZr_{0.51}Ti_{0.49}O_3$ (0.5 wt%: MnO_2) is shown in Fig. 3(a) and (b). The mechanical Q factor for $PbZr_{0.51}Ti_{0.49}O_3$(0.5 wt%: MnO_2) is 667. The agreement between the experimental and the calculated values are excellent. This shows that the present iterative method can be applied for versatile specimens without regarding the magnitude of Q. The complex materials constants for $PbZr_{0.51}Ti_{0.49}O_3$ (1 wt%: Nb_2O_5) and $PbZr_{0.51}Ti_{0.49}O_3$ with rather low Q values are given in Table 2 in addition to the values calculated by conventional R–A method.

The difference between the elastic compliance coefficients determined by R–A method and G- B fitting method is rather small (less than a few %) but the difference in the piezo-

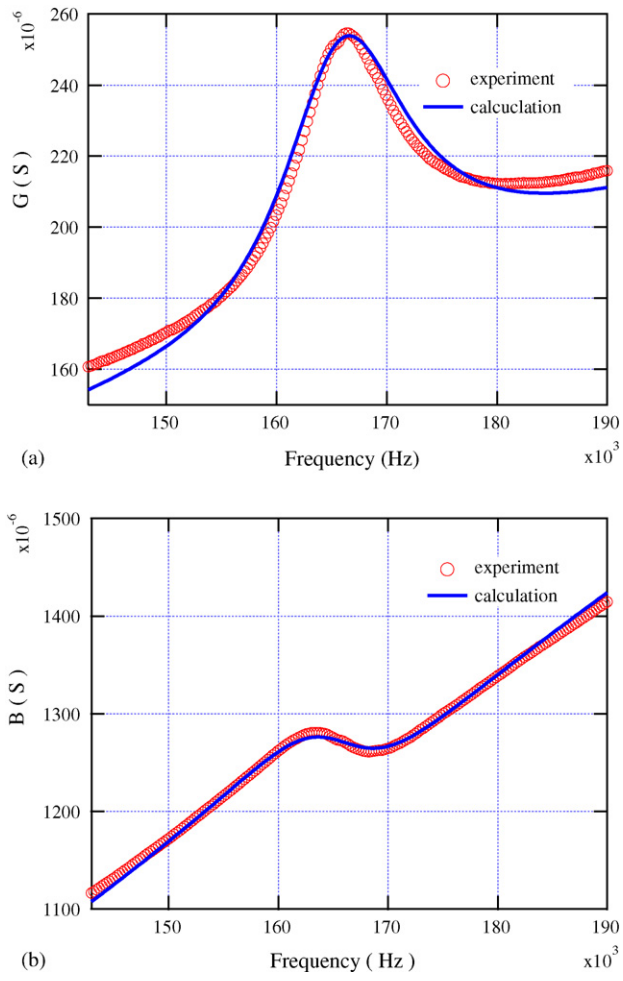

Fig. 2. (a) G (conductance) and (b) B (susceptance) of $K_{0.9}Li_{0.1}NbO_3$ in radial vibration mode.

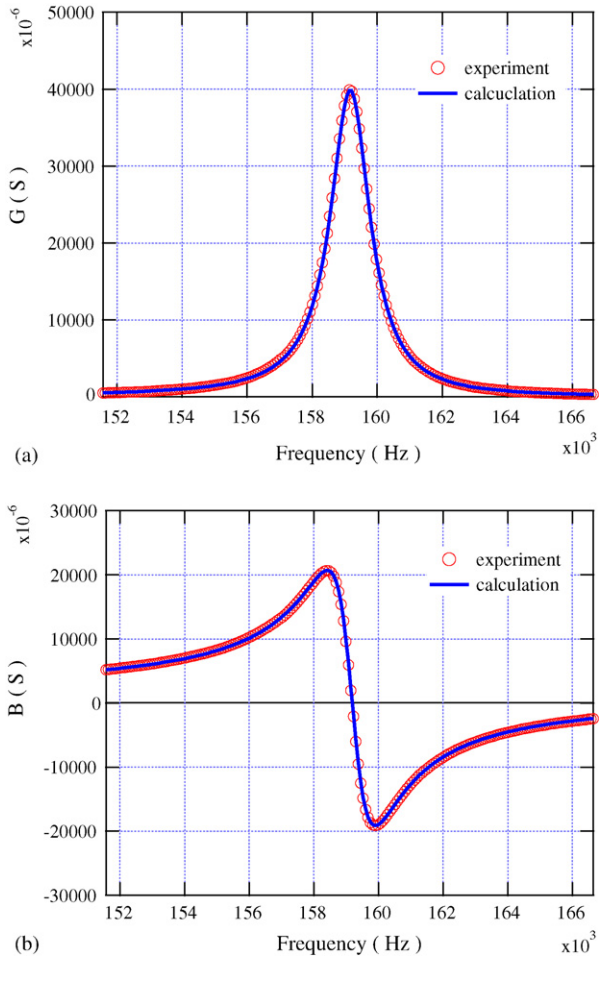

Fig. 3. (a) G (conductance) and (b) B (susceptance) of $PbZr_{0.51}Ti_{0.49}O_3$ (0.5 wt% MnO_2) in radial vibration mode.

electric constants and electromechanical coupling factor is large, especially for $K_{0.9}Li_{0.1}NbO_3$. The piezoelectric constant d_{31} determined by R–A method is 4.7 times larger than that determined by *G–B* fitting for $K_{0.9}Li_{0.1}NbO_3$. The electromechanical coupling factor determined by R–A method is 3.3 times larger than that obtained by *G–B* fitting method. The same tendency is found in $PbZr_{0.51}Ti_{0.49}O_3$ (1 wt%: Nb_2O_5) and $PbZr_{0.51}Ti_{0.49}O_3$. The piezoelectric constant and electromechanical coupling factor determined by R–A method for low Q materials are substantially higher than the values determined by the *G–B* fitting method.

5. Discussion

The IEEE Standard method has the advantages of being quick and not requiring any expensive equipment. These advantages, however, are accompanied by ignorance of all the phase

Table 1
Complex materials constants of $K_{0.9}Li_{0.1}NbO_3$ and $PbZr_{0.51}Ti_{0.49}O_3$ (0.5 wt% MnO_2) estimated by resonance–antiresonance (R–A) method and *G–B* fitting

	$K_{0.9}Li_{0.1}NbO_3$		$PbZr_{0.51}Ti_{0.49}O_3$ (0.5 wt% MnO_2)	
	R–A method	*G–B* fitting	R–A method	*G–B* fitting
$s_{11}^{E'}$ (m^2/N)	1.372×10^{-11}	1.533×10^{-11}	1.036×10^{-11}	1.034×10^{-11}
$s_{11}^{E''}$ (m^2/N)	3.536×10^{-13}	5.099×10^{-12}	1.474×10^{-14}	0.905×10^{-14}
d_{31}' (C/N)	-4.580×10^{-11}	-8.402×10^{-12}	-8.463×10^{-11}	-7.765×10^{-11}
d_{31}'' (C/N)		-3.136×10^{-12}		-1.658×10^{-13}
$\varepsilon_{33}^{T'}$ (F/m)	3.703×10^{-9}	1.874×10^{-9}	8.223×10^{-9}	7.647×10^{-9}
$\varepsilon_{33}^{T''}$ (F/m)		2.558×10^{-10}		7.085×10^{-10}
σ	0.34	0.33	0.33	0.33
k_p	0.26	0.08	0.56	0.48
Q	35.2	29	702	667

Table 2
Complex materials constants of $PbZ_{0.51}Ti_{0.49}O_3$ and $PbZr_{0.51}Ti_{0.49}O_3$ (1 wt%Nb_2O_5) estimated by resonance–antiresonance (R–A) method and *G–B* fitting

	$PbZ_{0.51}Ti_{0.49}O_3$		$PbZr_{0.51}Ti_{0.49}O_3$ (1 wt%Nb_2O_5)	
	R–A method	*G–B* fitting	R–A method	*G–B* fitting
$s_{11}^{E'}$ (m^2/N)	1.230×10^{-11}	1.192×10^{-11}	1.216×10^{-11}	1.112×10^{-11}
$s_{11}^{E''}$ (m^2/N)	9.517×10^{-14}	3.507×10^{-13}	1.185×10^{-13}	5.418×10^{-13}
d_{31}' (C/N)	-2.461×10^{-11}	-3.248×10^{-11}	-3.129×10^{-12}	-9.0666×10^{-11}
d_{31}'' (C/N)		-2.354×10^{-12}		-2.067×10^{-12}
$\varepsilon_{33}^{T'}$ (F/m)	8.153×10^{-9}	7.926×10^{-9}	1.049×10^{-8}	8.864×10^{-8}
$\varepsilon_{33}^{T''}$ (F/m)		5.610×10^{-11}		1.896×10^{-10}
σ	0.35	0.34	0.34	0.34
k_p	0.19	0.14	0.5	0.16
Q	129	119	103	87

information.[8] For the specimens with low Q the magnitude of imaginary parts of the elastic compliance coefficients and the piezoelectric constants are large, hence the apparent electromechanical coupling constant (k_p) obtained by R–A method is substantially higher than the actual value as seen in Table 1 for $K_{0.9}Li_{0.1}NbO_3$.

These disadvantages of the IEEE Standard method have been overcome by the present automated iterative method (*G–B* fitting) as mentioned in the previous paper. The formulation of the admittance for the resonator (Eq. (1)) does not have any approximation, thus if the iterative calculation reasonably converges the materials constants determined are accurate and reliable. In the present iterative method, all the information of the phase angles of the materials constants (dielectric constants, elastic compliance coefficients and piezoelectric constants) are taking into consideration. The voltage of measuring signal is less than 0.1 V, which is well below the threshold of nonlinearity even around the resonance frequency. The dielectric constants around the piezoelectric resonance frequency can be accurately determined in the course of the iterative fitting process. In the present method the data around the fundamental resonance is sufficient to perform iterative calculation except that the 1st overtone frequency $f_s^{(2)}$ is required to estimate the Poisson ratio σ.

It has been reported that the nonlinear iterative method has advantages among the other method especially for the specimens with low Q values.[9] Though the previous iterative methods for determination of the complex materials constants [2–4,10] are successful in the research work, however, they requires the frequencies (f_1 and f_2) in the peripheral iteration loop where the admittance data contains maximum piezoelectric energy.[3,4,10] The process for determining these frequencies involves some ambiguities, hence it is not feasible to automate the iteration process. In the present iterative method, the explicit procedure to estimate the appropriate initial values is proposed and the iteration process can be fully automated. The data of fundamental and 1st overtone frequencies and $\Delta f_s^{(1)}$ are required to estimate the initial s_{11}^E and Poisson's ratio σ. The frequency $f_s^{(1)}$ and the frequencies $f_{s1}^{(1)}$, $f_{s2}^{(1)}$ of the fundamental resonance, and G (conductance) and B (susceptance) at $f_{s1}^{(1)}$ and $f_{s2}^{(1)}$ are sufficient for determination of the initial values for the nonlinear iteration process.

In the present iterative method the Poisson's ratio was calculated by solving two transcendental equations (3) and (4) numerically, thus there is no requirements for tables or polynomial approximation formulas.[2,4] All the process has been automated in the present iterative method.

In the previous iterative method, though the data acquisition has been performed around the resonance frequency range, materials constants are determined only by the use of the frequencies f_s, f_p, f_1 and f_2 and the admittance $Y(f_1)$, $Y(f_2)$ and $Y(f_p)$.[3,4,10] The whole spectra around the resonance frequency have not been utilized. In the present iterative method, determination of complex materials constants has been achieved by a curve fitting approach. A modified from of the Levenberg–Marquardt nonlinear regression routine (LM routine) for complex admittance equation have been employed.[7] This allows for the determination of the best fit for an analytical curve to experimental results over any frequency range of interest. The duration of data acquisition for the frequency spectra of G and B around the fundamental resonance is less than 5 min and the iteration calculation converges only in a few seconds.

6. Conclusion

An iterative automated method for the characterization of piezoelectric specimens of disks in the radial vibration mode has been developed. The present nonlinear iteration method (*G–B* fitting method) has the advantages that a judicious choice of admittance data is not necessary and the complex materials constants are determined by fitting the admittance spectra around the fundamental resonance frequency to the theoretical expression of admittance.

The complex materials constants (dielectric constants, elastic compliance coefficients and piezoelectric constants) of piezoelectric specimens with moderate or low Q are accurately determined. The availability of the present *G–B* fitting method for the characterization of the specimens with very low Q values ($K_{0.9}Li_{0.1}NbO_3$) has been demonstrated. The duration for data acquisition is less than 5 min and the iteration calculation takes only a few seconds. The whole processing time for the *G–B* fitting method is rather short. These results show that the present iteration procedure (*G–B* fitting) will be an extremely useful tool for characterizing a variety of piezoelectric materials.

Appendix A

A.1. The linearized admittance Y_1

The linearized admittance Y_1 is given as follows:

$$Y_1 = G_1 + jB_1$$

where G_1 and B_1 represent the linearized conductance and susceptance, respectively:

$$\begin{aligned}
G_1 = {} & \omega\frac{\pi a^2}{t}\frac{4d'_{31}d''_{31}}{(1-\sigma)s_{11}^{E'}}\frac{(1+\sigma)J_1(z_0)}{z_0J_0(z_0)-(1-\sigma)J_1(z_0)} \\
& -\omega\frac{\pi a^2}{t}\frac{2d'^2_{31}}{(1-\sigma)}\frac{s_{11}^{E''}}{(s_{11}^{E'})^2}\frac{(1+\sigma)J_1(x_0)}{z_0J_0(z_0)-(1-\sigma)J_1(z_0)} \\
& +\omega\frac{\pi a^2}{t}\frac{d'^2_{31}}{(1-\sigma)s_{11}^{E'}}\frac{s_{11}^{E''}}{s_{11}^{E'}}\frac{(1+\sigma)[J_0(z_0)-(J_1(z_0)/z_0)]}{z_0J_0(z_0)-(1-\sigma)J_1(z_0)}z_0 \\
& -\omega\frac{\pi a^2}{t}\frac{d'^2_{31}}{(1-\sigma)s_{11}^{E'}}\frac{s_{11}^{E''}}{s_{11}^{E'}} \\
& \times\frac{\begin{array}{l}(1+\sigma)J_1(z_0)\{z_0J_1(z_0)-J_0(z_0)\\ +(1-\sigma)(J_0(z_0)-(J_1(z_0)/z_0))\}\end{array}}{\{z_0J_0(z_0)-(1-\sigma)J_1(z_0)\}^2}z_0 \\
& +\omega\frac{\pi a^2}{t}\varepsilon_{33}^{T''}-\omega\frac{\pi a^2}{t}\frac{4}{1-\sigma}\frac{d'_{31}d''_{31}}{s_{11}^{E'}} \\
& +\omega\frac{\pi a^2}{t}\frac{2}{1-\sigma}\frac{d'^2_{31}s_{11}^{E''}}{s_{11}^{E'2}}
\end{aligned}$$

$$\begin{aligned}
B_1 = {} & \omega\frac{\pi a^2}{t}\varepsilon_{33}^{T'}+\omega\frac{\pi a^2}{t}\frac{2}{(1-\sigma)s_{11}^{E'}} \\
& \times\left\{\frac{(1+\sigma)J_1(z_0)}{z_0J_0(z_0)-(1-\sigma)J_1(z_0)}-1\right\}d'^2_{31},
\end{aligned}$$

where

$$\omega = 2\pi f$$

and

$$z_0 = \omega a\sqrt{\rho s_{11}^{E'}(1-\sigma)}.$$

A.2. Matrix elements in Eq. (5)

The matrix elements in Eq. (5) are given as follows:

$$b_1(f) = \omega\frac{\pi a^2}{t},$$

$$b_2(f) = \omega\frac{\pi a^2}{t}\frac{2}{(1-\sigma)s_{11}^{E'}}\left\{\frac{(1+\sigma)J_1(z_0)}{z_0J_0(z_0)-(1-\sigma)J_1(z_0)}-1\right\}$$

A.3. Matrix elements of Eq. (6)

The matrix elements of Eq. (6) is given as follows:

$$g_1(f) = \omega\frac{\pi a^2}{t},$$

$$g_2(f) = \omega\frac{\pi a^2}{t}\frac{4d'_{31}}{(1-\sigma)s_{11}^{E'}}\left\{\frac{(1+\sigma)J_1(z_0)}{z_0J_0(z_0)-(1-\sigma)J_1(z_0)}-1\right\},$$

$$\begin{aligned}
g_3(f) = {} & \omega\frac{\pi a^2}{t}\frac{d'^2_{31}}{(1-\sigma)}\frac{s_{11}^{E''}}{s_{11}^{E'2}}\left\{-2\frac{(1+\sigma)J_1(z_0)}{z_0J_0(z_0)-(1-\sigma)J_1(z_0)}\right. \\
& +\frac{(1+\sigma)[J_0(z_0)-((J_1(z_0))/(z_0))]}{z_0J_0(z_0)-(1-\sigma)J_1(z_0)}z_0 \\
& \left.\times\frac{\begin{array}{l}(1+\sigma)J_1(z_0)\{d''_{31}J_1(x_0)-J_0(x_0)\\ +(1-\sigma)(J_0(x_0)-((J_1(z_0))/(z_0)))\}\end{array}}{\{z_0J_0(z_0)-(1-\sigma)J_1(z_0)\}^2}z_0+2\right\}
\end{aligned}$$

References

[1]. Maeda, M., Hashimoto, H. and Suzuki, I., Measurements of complex materials constants of piezoelelctrics: extentional vibrational mode of a rectangular bar. *J. Phys. D: Appl. Phys.*, 2003, **36**, 176–180.

[2]. Sherrit, S., Gauthier, N., Woederick, H. D. and Mukherjee, B. K., Ferroelectrics accurate evaluation of the real and imaginary constants for a piezoelelctric resonator in the radial mode. 1991, **119**, 17–32.

[3]. Alemany, C., Gonzaelz, A. M., Pardo, L., Jimmenez, B., Carmona, F. and Mendiola, J., Automatic determination of complex constants of piezoelectic lossy materials in the radial mode. *J. Phys. D: Appl. Phys.*, 1994, **27**, 148–155.

[4]. Amarande, L., Miclea, C. and Tanasoiu, C., Iterative evaluation of the complex constants of piezoceramic resonator in the radial mode. *J. Eur. Ceram. Soc.*, 2002, **22**, 1873–1881.

[5]. Meitzler, A. H., O'Brayan, H. M. and Tiersten Jr., H. F., Definition and measurement of radial mode coupling factors in piezoelelctric ceramic materials with large variations in poisson's ratio. *IEEE Trans. Sonics Ultrasonics*, 1973, **20**, 233–239.

[6]. Holland, R., Representation of dielelctric, elastic, and piezoelelctric losses by complex coefficients. *IEEE Trans. Sonics Ultrasonics*, 1967, **14**, 18–20.

[7]. Press, W. H., Teukolsky, S. A., Vetterling, W. T. and Flannery, B. P., Numerical Recipes in C. 2nd ed Cambridge University Press, Cambridge, 1992, p. 681.

[8]. IEEE Standard on Piezoelectricity ANSI/IEEE Std 176-1987, The Institute of Electrical and Electronics Engineers, New York, 1987.

[9]. Kwok, K. W., Chan, H. L. W. and Choy, C. L., Evaluation of the material parameters of piezoelectric materials by various method. *IEEE Trans. Ultrasonics, Ferroelectr. Freq. Contr.*, 1997, **44**, 733–742.

[10]. Smits, J. G., Influence of moving domain walls and jumping lattice defects on complex material coefficients of piezoelelctrics. *IEEE Trans. Sonics Ultrasonics*, 1976, **23**, 393–402.